Bayes Filtering

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1 Introduction

2 Sources

- `https://medium.com/@vikramsetty169/the-bayes-filter-71f8b61afc1c`
- `https://www.cs.cmu.edu/~16831-f14/notes/F14/16831_lecture02_prayana_tdecker_humphreh.pdf`

3 Derivation of Update Rule

As shown in Fig 1, we assume a Hidden Markov Model for the problem.

Let the action be U_i, the hidden state be X_i, and the observation as Z. Define a belief as

$$B(X_t) = \Pr(X_t|Z_1...Z_t, U_1...U_t)$$

In English, this is the probability of a particular hidden state given a set of observations and actions.

Use Bayes Rule

$$\begin{aligned} &\Pr(X_t|Z_1...Z_t, U_1...U_t) \cdot \Pr(Z_t|Z_1...Z_{t-1}, U_1...U_t) \\ =&\Pr(Z_t|X_t, Z_1...Z_{t-1}, U_1...U_T) \cdot \Pr(X_t|Z_1...Z_{t-1}, U_1...U_t) \end{aligned}$$

This is easier to see if we hide the remaining variables with $... := Z_1...Z_{t-1}, U_1...U_t$.

$$\begin{aligned} &\Pr(X_t|Z_t, ...) \cdot \Pr(Z_t|...) \\ =&\Pr(Z_t|X_t, ...) \cdot \Pr(X_t|...) \quad Pr(X_t|Z_t...) = B(X_t) \end{aligned}$$

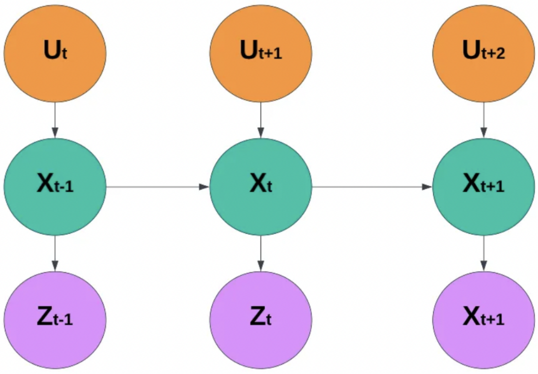

Figure 1: Setup of Problem, from `https://medium.com/@vikramsetty169/the-bayes-filter-71f8b61afc1c`

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So then, the belief is:

$$B(X_t) = \frac{Z_t|X_t, ...) \cdot \Pr(X_t|...)}{\Pr(Z_t|...)}$$

We can use the Markov assumption that Z_t is independent of $Z_1...Z_{t-1}, U_1...U_t)$ (i.e. observations are independent of past observations and actions). so $\frac{1}{Pr(Z_t|Z_1...Z_{t-1}, U_1...U_t} = \eta$ is constant.

Then,

$$B(X_t) = \eta \Pr(Z_t|X_t) \cdot \Pr(X_t|...)$$

From the law of total probability,

$$\Pr(A|C) = \sum_B (\Pr(A|B, C) \Pr(B|C)$$

or for continuous distributions:

$$\Pr(A|C) = \int (\Pr(A|B, C) \Pr(B|C) dB$$

So we can expand $B(X_t)$ to:

$$B(X_t) = \eta \Pr(Z_t|X_t) \cdot \int \Pr(X_t|X_{t-1}, ...) \Pr(X_{t-1}|...) dX_{t-1}$$

Recall that $... := Z_1...Z_{t-1}, U_1...U_t$. By the Markov assumption, X_{t-1} does not depend on U_t and X_t only depends on U_t and X_{t-1}.

$$B(X_t) = \eta \Pr(Z_t|X_t) \cdot \int \Pr(X_t|X_{t-1}, U_t) \Pr(X_{t-1}|(Z_1...Z_{t-1}, U_1...U_{t-1})dX_{t-1}$$

Finally, using the definition of the belief for $B(X_{t-1})$:

$$B(X_t) = \eta \Pr(Z_t|X_t) \cdot \int \Pr(X_t|X_{t-1}, U_t)B(X_{t-1})dX_{t-1} \quad \square$$

This is the Bayes filter.